

ADJUSTABLE BOUNDARY CONDITIONS FOR MULTIDIMENSIONAL TRANSFER FUNCTION MODELS

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ABSTRACT

Block based physical modeling requires to provide a library of modeling blocks for standard components of real or virtual musical instruments. Complex synthesis models are built by connecting standard components in a physically meaningful way. These connections are investigated for modeling a resonating structure as a distributed parameter system. The dependence of a resonator's spectral structure on the termination of its ports is analyzed. It is shown that the boundary conditions of a distributed parameter system can be adjusted by proper termination only. Examples show the corresponding variation of the resonator's spectral structure in response to variations of the external termination.

1. INTRODUCTION

1.1. Block Based Physical Modeling

There is a rich set of tools available for digital sound synthesis: wavetable synthesis, frequency modulation (FM), additive and subtractive synthesis, granular and concatenative synthesis, and various flavors of physical modeling. Rather than generating more and more new synthesis methods, recent advances have focused on the combination of different synthesis methods. In the context of physical modeling, a methodology for the block-wise synthesis of virtual musical instruments has been developed under the name of *block based physical modeling*.

Block based modeling separates the tasks of *component design* and *model building*. Component design means that various components of real or virtual instruments like strings, membranes, air columns, piano hammers, mallets, etc. are modeled and implemented independently of each other. The resulting component models (the *blocks*) are stored in a block library for later use. Model building means to build a virtual instrument from its components by selecting the appropriate blocks from the library and to connect them in a meaningful way. An overview on methods and synthesis tools for block based modeling can be found in [1] and the literature cited there; a detailed account of the fundamentals is given in [2].

1.2. Signals and Ports

This procedure is well known from signal based simulation environments like SIMULINK or programming languages for audio signals like Pure Data (PD). The block structure in these implementations resembles signal flow graphs known from systems and control theory. Blocks for processing signals have well defined inputs and outputs. The connection of the output of one block to the input of the next one does not change the values of the output

signal. This property is easy to implement in software but it is also shared by specialized hardware like analog modular synthesizers. In detail, electronic circuitry with low output impedance and high input impedance ensures that the output signal is not affected by connections to a limited number of inputs.

The situation is different when the blocks model physical components. At first, the related quantities (e.g. pressure and particle velocity in a pipe) are not per se given as input or output signals. Furthermore, connecting two blocks will affect all related quantities. This situation is usually described by so called ports, a combination of two or more variables like pressure and flow, force and velocity, or voltage and current. Connecting two physical modeling blocks means to connect the respective port variables, which in turn will change the behavior of both blocks.

1.3. Boundary Conditions

Designing blocks for physical modeling frequently requires to consider distributed parameter systems like strings, membranes, and air columns. Their implementation is based on a mathematical description in the form of partial differential equations (PDEs) and their respective boundary conditions. In musical instruments, boundary conditions are given e.g. by the fixing of a string, a membrane, or a plate, or by the termination of an air column. The type of boundary conditions determines the sound of a resonating structure, as is well known from string, brass, and woodwind instruments or from organ pipes.

Boundary conditions of distributed parameter systems are closely related to the port variables of their block implementations. In short, the port variables are the values of the block model at the interface to the outside world, i.e. to other block models. Conditions on the port variables imposed by block connections or terminations constitute the boundary conditions for the distributed parameter block. Examples are the excitation of a string, which is zero at a fixed end or the pressure in a pipe which is zero at an open end.

The mathematical literature classifies boundary conditions of the first, second, and third kind [3, 4]. Boundary conditions of the first and second kind prescribe the values of port variables or their derivatives. Boundary conditions of the third kind prescribe relations between the port variables. These relations may be real or complex valued and are given in terms of reflection factors or impedances. Methods for the investigation of resonance modes in a one-dimensional medium with two resistive boundaries have been compared in [5].

1.4. Connecting Blocks

At this point, the separation of component design and model building discussed above poses a problem which is the topic of this contribution: During component design, i.e. when a distributed parameter model is implemented, the boundary conditions for the use of this component for model building are not known. Moreover, a certain block has to work in a physically meaningful way in different kinds of connections.

However mathematical rigor requires that the boundary conditions are included in the definition of a distributed parameter model in order to constitute a properly posed problem. This means that a distributed parameter block at first has to be designed and implemented for a certain set of boundary conditions and later used in block connections which impose other boundary conditions.

This problem can also be expressed in musical terms. When a block model of e.g. a string with fixed ends is connected with another block, e.g. a sound board, will the spectral structure of the block model change accordingly? Is it sufficient to provide the correct port connections to the existing blocks or is it necessary to redesign the string model?

This problem is discussed here for a specific case. A block model for an air column with standard boundary conditions is terminated by an external component and the resulting spectral properties are investigated. The answer to the question above is given by formulating the problem as a feedback structure and by analyzing it in terms of basic control theory.

2. PROBLEM DESCRIPTION

This section describes the problem in general terms. Block models of distributed parameter systems are introduced, the boundary conditions are formulated, and an example for wave propagation is presented.

2.1. Block Models of Distributed Parameter Systems

A general distributed parameter system with one spatial dimension is shown in Fig. 1. It may represent a vibrating string, an air column, or another type of waveguide. The spatial coordinate is denoted by x , the model is defined within the one-dimensional spatial region $V = [0, l]$ with the boundary $\partial V = \{0; l\}$. For all boundary points $x_b \in \partial V$, i.e. $x_b \in \{0; l\}$, the behavior is determined by two physical variables $y_1(x_b)$ and $y_2(x_b)$. They constitute the port variables introduced above. These variables may represent force and deflection, pressure and particle velocity, or other pairs of across and through variables, depending on the nature of the distributed system. Two of these variables are sufficient to describe simple resonating structures. More involved models with more than two variables can be investigated in the same way, but they are not discussed here.

The internal behavior of the system in Fig. 1 is described in terms of a vector partial differential equation (1). The vector $\mathbf{y}(x, t)$ consists of the two variables $y_1(x, t)$ and $y_2(x, t)$, the vector $\mathbf{v}(x, t)$ describes a possible excitation function. The matrices \mathbf{B}_1 and \mathbf{B}_2 describe the partial differential operators in detail. An example is given in Sec. 2.3.

$$\left[\mathbf{B}_1 \frac{\partial}{\partial x} + \mathbf{B}_2 \frac{\partial}{\partial t} \right] \mathbf{y}(x, t) = \mathbf{v}(x, t), \quad x \in V \quad (1)$$

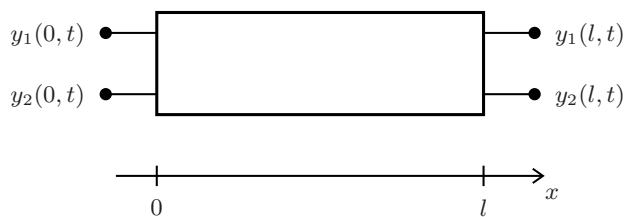


Figure 1: Sketch of a spatially one-dimensional block model. The boundary consists of two points $x_b \in \{0; l\}$.

The port variables, i.e. the the outcome at the boundaries $\mathbf{y}(x, t)$ for $x = x_b$ are given by

$$\mathbf{y}(0, t) = \begin{pmatrix} y_1(0, t) \\ y_2(0, t) \end{pmatrix}, \quad \mathbf{y}(l, t) = \begin{pmatrix} y_1(l, t) \\ y_2(l, t) \end{pmatrix}. \quad (2)$$

2.2. Boundary Behavior

The port variables are neither inputs nor outputs in the sense that e.g. $y_1(x_b)$ is independent of $y_2(x_b)$ and $y_2(x_b)$ is determined only by $y_1(x_b)$. If the port variables $\mathbf{y}(x, t)$ for $x = x_b$ are connected to the port variables of another block, then their values are determined by the interaction of both blocks.

This interaction happens instantly for continuous-time systems. For discrete-time systems, it is necessary to ensure computability by avoiding delay-free loops. To this end, the elements of $\mathbf{y}(x_b, t)$ or combinations thereof have to be divided into input and output variables. The formal description of this division requires to introduce the normal component of the differential operator (see [2])

$$\mathbf{B}_n = n_1 \mathbf{B}_1 + n_2 \mathbf{B}_2 \quad (3)$$

with the normal vector

$$\mathbf{n}_b = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}. \quad (4)$$

This notation allows for rather general boundary conditions including time-varying boundaries. For the consideration of boundary conditions at time-invariant boundary points, \mathbf{n}_b has the values

$$\mathbf{n}_b = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{for } x = 0, \quad \mathbf{n}_b = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } x = l, \quad (5)$$

such that

$$\mathbf{B}_n = \pm \mathbf{B}_1. \quad (6)$$

Now the boundary inputs and boundary outputs can be defined by introducing the boundary input operator \mathbf{f}_b and the boundary output operator \mathbf{f}_o , both are two-element column vectors. They define the input signal $v_b(x_b)$ and the output signal $y_b(x_b)$ at the boundary x_b as

$$\mathbf{f}_b^H \mathbf{B}_n \mathbf{y}_b(x_b) = v_b(x_b) \quad \text{input}, \quad (7a)$$

$$\mathbf{f}_o^H \mathbf{B}_n \mathbf{y}_b(x_b) = y_b(x_b) \quad \text{output}. \quad (7b)$$

The superscript H denotes the Hermitian vector or matrix. For real valued boundary operators, it is equal to the transposed vector.

The boundary input operator \mathbf{f}_b and the boundary output operator \mathbf{f}_o can be combined to a matrix representation to replace (7a,7b) by

$$\left(\mathbf{f}_b \ ; \ \mathbf{f}_o \right)^H \mathbf{B}_n \mathbf{y}(x_b) = \begin{pmatrix} v_b(x_b) \\ y_b(x_b) \end{pmatrix}. \quad (8)$$

This definition of input and output values is only meaningful if $v_b(x_b)$ and $y_b(x_b)$ are not identical, i.e. if

$$\text{rank} \left\{ \left(\mathbf{f}_b \ ; \ \mathbf{f}_o \right)^H \mathbf{B}_n \right\} = \text{rank} \{ \mathbf{B}_n \}. \quad (9)$$

Fig. 2 shows the relation (8) between the port variables at $x = l$ and the input and output signals $v_b(l, t)$ and $y_b(l, t)$. The assignment between the port variables and the input and output signals is defined by the boundary input operator \mathbf{f}_b and the boundary output operator \mathbf{f}_o . Only the port at $x = l$ is considered here. Similar results hold also for the port at $x = 0$ with the appropriate normal vector from (5).

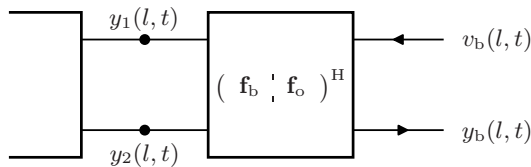


Figure 2: Spatial one-dimensional model with a specific boundary behavior described by equation (8).

2.3. Example: Wave Equation

As an example serves an air column with sound pressure $p = p(x, t)$, particle velocity $v = v(x, t)$, mass density ρ_0 , and speed of sound c . A simple distributed parameter model is given by

$$\left[\begin{matrix} \mathbf{I}_0 \\ \mathbf{B}_1 \end{matrix} \frac{\partial}{\partial x} + \underbrace{\begin{pmatrix} 0 & -1 \\ -1/c^2 & 0 \end{pmatrix}}_{\mathbf{B}_2} \frac{\partial}{\partial t} \right] \underbrace{\begin{pmatrix} p \\ -\rho_0 v \end{pmatrix}}_{\mathbf{y}} = \underbrace{\begin{pmatrix} 0 \\ -f_e \end{pmatrix}}_{\mathbf{v}} \quad (10)$$

where the underbraces indicate the corresponding vectors and matrices of the general system (1).

The normal component of the differential operator according to (6) is given by

$$\mathbf{B}_n = \pm \mathbf{B}_1 = \pm \mathbf{I}_0, \quad (11)$$

where \mathbf{I}_0 denotes the identity matrix.

Meaningful boundary input and output operators have to satisfy the rank condition

$$\text{rank} \left\{ \left(\mathbf{f}_b \ ; \ \mathbf{f}_o \right) \right\} = \text{rank} \{ \mathbf{I}_0 \} = 2 \quad (12)$$

If the case of hard reflecting walls on both sides (i.e. at $x = 0$ and $x = l$) is considered as boundary condition, then the particle velocity at the boundary must be zero. If (10) is regarded as a PDE in terms of the sound pressure then these boundary conditions are of the second kind (Neumann boundary conditions). This assumption implies together with the rank condition (12)

$$\left(\mathbf{f}_{b2} \ ; \ \mathbf{f}_{o2} \right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (13)$$

and thus the following assignment between port variables and the input and output signals

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(l, t) \\ y_2(l, t) \end{pmatrix} = \begin{pmatrix} v_{b2}(l, t) \\ y_{b2}(l, t) \end{pmatrix}. \quad (14)$$

The subscripts b2 and o2 denote boundary conditions of the second kind. The relation (14) is shown graphically in Fig. 3 by specializing the boundary operators from Fig. 2 to (13). Another type of boundary conditions is considered in the following section.

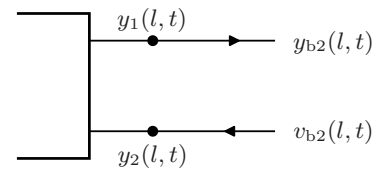


Figure 3: Block model with boundary conditions of the second kind.

3. ADJUSTABLE BOUNDARY CONDITIONS

This section describes a different kind of boundary conditions which include adjustable parameters. If not connected to another block, this kind of boundary conditions is equivalent to a port termination.

3.1. Boundary Conditions of the Third Kind

The assignment of boundary conditions of the second kind (14) declares one port signal ($y_1(l, t)$) to be the output ($y_{b2}(l, t)$) and the other port signal ($y_2(l, t)$) to be the input ($v_{b2}(l, t)$). But it is also possible to declare a certain linear combination of the port signals to be the output and another linear combination to be the input. Such an assignment corresponds to boundary conditions of the third kind (also called Robin's boundary conditions) and is given by

$$\left(\mathbf{f}_b \ ; \ \mathbf{f}_o \right) = \begin{pmatrix} 1 & g_2 \\ g_1^{-1} & 1 \end{pmatrix} \quad \text{with } g_1 \neq g_2, \quad (15)$$

where g_1 and g_2 are real admittances. Their physical dimensions have to be compatible with the port variables $y_1(l, t)$ and $y_2(l, t)$. The rank condition (12) requires $g_1 \neq g_2$.

The input and output variables $v_{b3}(l, t)$ and $y_{b3}(l, t)$ are assigned by

$$\begin{pmatrix} 1 & g_1^{-1} \\ g_2 & 1 \end{pmatrix} \begin{pmatrix} y_1(l, t) \\ y_2(l, t) \end{pmatrix} = \begin{pmatrix} v_{b3}(l, t) \\ y_{b3}(l, t) \end{pmatrix}. \quad (16)$$

The subscripts b3 and o3 denote boundary conditions of the third kind.

Due to the definition of input and output variables by (16), the port variables $y_1(l, t)$ and $y_2(l, t)$ themselves are not computed in a specific order. However, to establish a relation with boundary conditions of the second kind, now consider arbitrarily $y_1(l, t)$ as an output variable of the port at $x = l$. To realize the boundary block from Figure 2, one has to solve for $y_2(l, t)$ and $y_{b3}(l, t)$

$$y_2(l, t) = g_1 \cdot v_{b3}(l, t) - g_1 \cdot y_1(l, t), \quad (17a)$$

$$y_{b3}(l, t) = 1 \cdot y_2(l, t) + g_2 \cdot y_1(l, t). \quad (17b)$$

Fig. 4 shows the lattice structure of the corresponding signal flow graph.

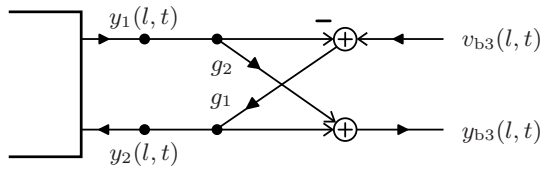


Figure 4: Block model with boundary conditions of the third kind.

3.2. Port Termination

Here no connection to other blocks is considered and therefore the input signal $v_{b3}(l, t)$ is zero and the output signal $y_{b3}(l, t)$ is not required. Then the realization of boundary conditions of the third kind simplifies to a termination of the port at $x = l$ with boundary conditions of the second kind (see Fig.3) by a negative admittance g_1 as shown in Fig. 5. Adjusting the coefficient $-g_1$ changes the character of this termination.

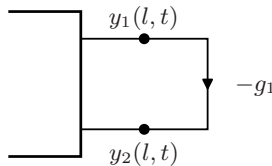


Figure 5: Simplified boundary conditions, the boundary input is zero and the boundary output is not required.

3.3. Realization of Adjustable Boundary Conditions

Based on the results of Sec. 3.1 and 3.2, adjustable boundary conditions with a parameter g_1 may be realized by block based physical modeling as follows:

- Design a standard block model according to Sec. 2.3 with boundary conditions of second kind. According to Fig. 2 one port variable is the input variable and the other one is the output variable. This block can be designed during component design and stored in a block library for later use.
- For realization of adjustable boundary conditions during model building, use the previously designed block and apply an external termination to its port at $x = l$. It consists of a feedback of the scaled port output back to the port input. The scaling multiplier may be any real number. According to Fig. 4 and Fig. 5 this procedure realizes boundary conditions of the third kind from an existing block model.

4. INTERPRETATION

Although the results of the previous section follow directly from the boundary conditions from equation (17), they do not explain the resulting change in the spectral structure of the initial block model. This section gives an interpretation of the previous results based on tools from basic control theory, i.e. transfer function formulation and feedback analysis.

4.1. Transfer Function Formulation

Now apply the procedure from Sec. 3.3 for a discrete-time block model:

- design a discrete-time model with boundary conditions of second kind.
- turn it into a model with boundary conditions of third kind by termination with a suitable admittance.

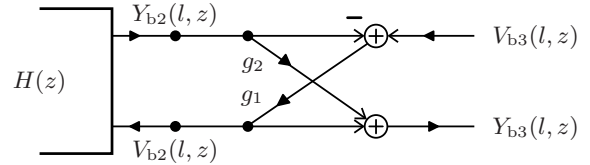


Figure 6: Block model with boundary conditions of third kind.

Fig. 6 shows the discrete-time version of Fig. 4. $V_{b2}(l, z)$ and $Y_{b2}(l, z)$ are z -transforms of discrete-time approximations $v_{b2}[l, k]$ and $y_{b2}[l, k]$ of their continuous counterparts $v_{b2}(l, t)$ and $y_{b2}(l, t)$. Corresponding relations hold for $V_{b3}(l, z)$ and $Y_{b3}(l, z)$. $H(z)$ is the transfer function of a discrete approximation of the distributed system with boundary conditions of the second kind with respect to its port variables, i.e.

$$H(z) = \frac{Y_{b2}(l, z)}{V_{b2}(l, z)} = \frac{N(z)}{D(z)}. \quad (18)$$

$N(z)$ and $D(z)$ denote the numerator and the denominator polynomial, respectively.

The form of $H(z)$ depends on the kind of discrete-time approximation of the distributed system. An example for a realization with the functional transformation method is given in Section 4.2. Other physical modeling methods like e.g. the digital waveguide method [6, 7] yield different transfer functions with similar behavior.

4.2. Example: Functional Transformation Method

The functional transformation method (FTM) is used for physical modeling digital sound synthesis of resonating structures like strings, bars, air columns, membranes, plates, and alike. It starts from a PDE description of the continuous-time, continuous-space model and derives a discrete-time model in the form of a parallel arrangement of simple transfer functions. More details on the procedure and examples for the use of multidimensional transfer function models can be found in [2, 8].

Here it is sufficient to show the complex representation of a typical configuration in Fig. 7. It consists of a parallel arrangement of first order systems with complex feedback coefficient a_μ and further multipliers b_μ and c_μ derived from certain eigenvalue problems [2, 8]. These eigenvalue problems (so-called Sturm-Liouville problems) consider the boundary conditions of the system, here boundary conditions of the second kind.

The transfer function of a single first order system is

$$H_\mu(z) = b_\mu c_\mu \frac{z}{z - a_\mu} = \frac{d_\mu z}{z - a_\mu} \quad \text{with } d_\mu = b_\mu c_\mu. \quad (19)$$

The complete transfer function is given by the sum of all N first order systems from (19) as (see Fig. 7)

$$H(z) = \sum_{\mu=1}^N H_\mu(z) = \sum_{\mu=1}^N \frac{d_\mu z}{z - a_\mu}. \quad (20)$$

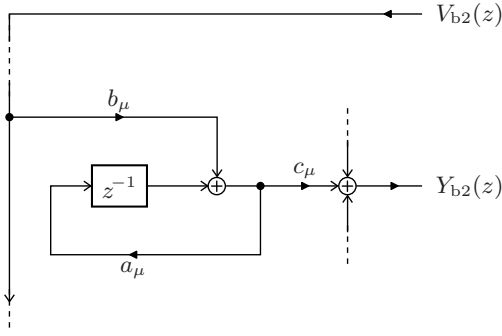


Figure 7: Structure of the discrete-time implementation achieved with the FTM, see for instance [8]. A finite number of N first order transfer functions with complex feedback coefficient a_{μ} are arranged in parallel.

The denominator $D(z)$ of (20) is the product of the denominators $z - a_{\mu}$ of the first order systems (19)

$$D(z) = \prod_{\nu=1}^N (z - a_{\nu}) . \quad (21)$$

The transfer function $H(z)$ from (20) turns with

$$D_{\mu}(z) = \frac{D(z)}{(z - a_{\mu})} = \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^N (z - a_{\nu}) \quad (22)$$

into

$$H(z) = \sum_{\mu} \frac{d_{\mu} z D_{\mu}(z)}{(z - a_{\mu}) D_{\mu}(z)} = \frac{N(z)}{D(z)} , \quad (23)$$

with the numerator

$$N(z) = z \sum_{\mu} d_{\mu} D_{\mu}(z) . \quad (24)$$

4.3. Feedback Analysis

In Sec. 3 it has been shown that boundary conditions of the third kind can be realized by external termination of a system with boundary conditions of the second kind. Now the relation between the transfer functions

$$H(z) = \frac{Y_{b2}(l, z)}{V_{b2}(l, z)} = \frac{N(z)}{D(z)} , \quad H_{b3}(z) = \frac{Y_{b3}(l, z)}{V_{b3}(l, z)} = \frac{N_{b3}(z)}{D_{b3}(z)} . \quad (25)$$

is derived, where $V_{b2}(l, z)$, $Y_{b2}(l, z)$ and $V_{b3}(l, z)$, $Y_{b3}(l, z)$ correspond to Fig. 6. Contrary to the example in Sec. 4.2 no specific implementation of $H(z)$ is assumed.

The lattice structure from Fig. 6 is represented in matrix notation by (see (14) and (16))

$$\begin{pmatrix} 1 & g_1^{-1} \\ g_2 & 1 \end{pmatrix} \begin{pmatrix} Y_{b2}(l, z) \\ V_{b2}(l, z) \end{pmatrix} = \begin{pmatrix} Y_{b3}(l, z) \\ V_{b3}(l, z) \end{pmatrix} \quad (26)$$

or

$$Y_{b2}(l, z) + g_1^{-1} V_{b2}(l, z) = V_{b3}(l, z) \quad (27)$$

$$g_2 Y_{b2}(l, z) + V_{b2}(l, z) = Y_{b3}(l, z) . \quad (28)$$

The transfer function $H_{b3}(z)$ follows from the division of (28) by (27) as (some arguments are omitted for convenience)

$$H_{b3}(z) = \frac{Y_{b3}}{V_{b3}} = \frac{g_2 Y_{b2} + V_{b2}}{Y_{b2} + g_1^{-1} V_{b2}} . \quad (29)$$

Dividing by V_{b2} and using (18) gives

$$H_{b3}(z) = \frac{g_2 H(z) + 1}{H(z) + g_1^{-1}} = \frac{g_2 \frac{N(z)}{D(z)} + 1}{\frac{N(z)}{D(z)} + g_1^{-1}} . \quad (30)$$

Multiplying by $D(z)$ results in

$$H_{b3}(z) = \frac{g_2 N(z) + D(z)}{N(z) + g_1^{-1} D(z)} . \quad (31)$$

Multiplication with g_1 finally gives

$$H_{b3}(z) = g_1 \frac{D(z) + g_2 N(z)}{D(z) + g_1 N(z)} = \frac{N_{b3}(z)}{D_{b3}(z)} . \quad (32)$$

Thus the transfer function for boundary conditions of the third kind $H_{b3}(z)$ is expressed by the numerator $N(z)$ and denominator $D(z)$ of the transfer function for boundary conditions of the second kind.

Investigating the denominator of $H_{b3}(z)$ yields the interpretation of the spectral effect of the external termination. Obviously, the resonances of $H_{b3}(z)$ are given by the poles of the denominator polynomial $D_{b3}(z)$

$$D_{b3}(z) = D(z) + g_1 N(z) . \quad (33)$$

Therefore the real coefficient g_1 allows to shift the pole locations according to the boundary conditions of third kind in equation (15). The effect of the external termination is shown by examples in the next section.

5. RESULTS

For a more intuitive illustration of the results from the previous sections, the modeling scenario as depicted in Fig. 8 is applied. The underlying PDE is the wave equation as given in (10). The input of the model is $y_2(0, t)$ while it is solved for the output variable $y_1(l, t)$. According to the definition of the vectorial outcome $\mathbf{y}(x, t)$ in (10) this corresponds to the particle velocity at the left side as the input and the pressure at the right side for output.

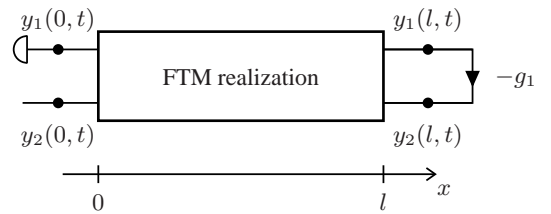


Figure 8: Sketch of the complete spatially one-dimensional block model. The boundary consists of two points $x_b \in \{0; l\}$.

The boundary conditions at the left side are of the second kind, while the boundary conditions on the right side are adjustable via the parameter g_1 . As well known from basic acoustics [9, 10],

boundary conditions of third kind (as given in (15)) directly correspond to a specific reflection factor α . In this scenario, this factor can be calculated by

$$\alpha = \frac{1 + cg_1}{1 - cg_1}, \quad (34)$$

where c is the speed of sound in the medium. A discrete realization of the model is achieved with the FTM (see [8] for instance). The sampling period T is chosen to result in $N = 17$ first order systems, one with a_μ equal to zero and eight pairs of complex conjugate systems. The resulting discrete time transfer function

$$G(z) = \frac{Y_1(l, z)}{Y_2(0, z)} \quad (35)$$

is depicted in Fig. 9 for four different reflection coefficients. The transfer function in Fig. 9(a) represents boundary conditions of the second kind at $x = l$. The model obviously represents a comb filter what is the expected behavior, as the traveling waves are perfectly reflected at all boundaries. The other extremum is depicted in Fig. 9(d), where the reflection coefficient α is zero. The right side absorbs all incoming waves as good as possible, such that the impulse response from $y_2(0, t)$ to $y_1(l, t)$ is a simple Dirac impulse. The de facto transfer function in Fig. 9(d) however is not constant for all frequencies, as only 17 modes are considered in the simulation. Figs. 9(b) and 9(c) show intermediate cases for $\alpha = 0.8$ and $\alpha = 0.5$.

Obviously there is a gradual variation of the spectral properties of the system in Fig. 8, although the air column model itself does not change. The variations of the transfer function $G(e^{j\Omega})$ are only caused by adjusting the feedback coefficient g_1 .

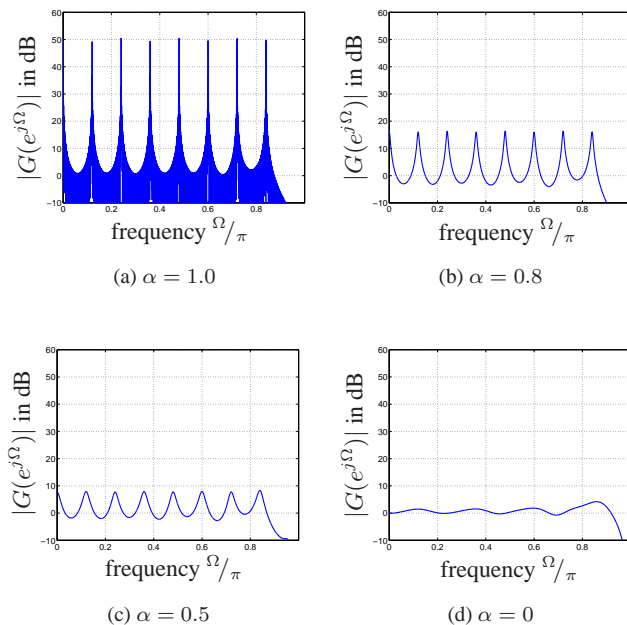


Figure 9: Normalized transfer function $G(z)$ from (34) evaluated at the unit circle $z = e^{j\Omega}$. The transfer function is depicted for different reflection coefficients α , which directly result from the feedback coefficient g_1 through (35).

6. CONCLUSIONS

The consideration of a simple example showed that models with adjustable boundary conditions can be built from multidimensional transfer functions with fixed boundary conditions. This result is important for the practical application of physical modeling. It is feasible to design components of musical instruments from the physical description of resonating structures. Although their implementation requires the assumption of fixed boundary conditions for a correct mathematical description, the spectral properties can be adjusted through proper port termination or, more general, by suitable connection to other modeling blocks.

These results have been obtained by considering one-dimensional transfer functions between suitably chosen port variables. The technique of calculating the denominator polynomial of a feedback structure from the corresponding open-loop transfer function is well-known in control theory as the root locus method. Considering the modeling block of a resonating structure with fixed boundary conditions as an open loop system (in control terms the *plant*), it is possible to adjust the properties of the closed loop system only by variations in the feedback path (the *controller*).

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